Set estimation with alphahull and alphashape3d

V Xornada de usuarios R en Galicia

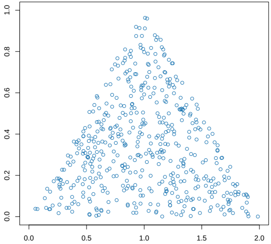
Beatriz Pateiro López

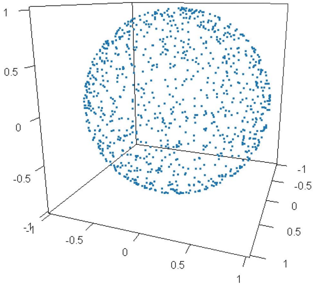
October 25th, 2018

Set estimation

Point sets are a fundamental source of information in many disciplines:

- samples from probability distributions





Set estimation

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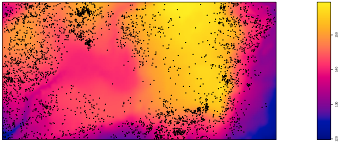
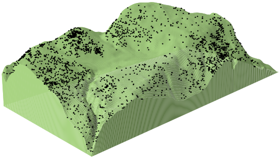
- portions of realizations of point processes observed within bounded sampling windows

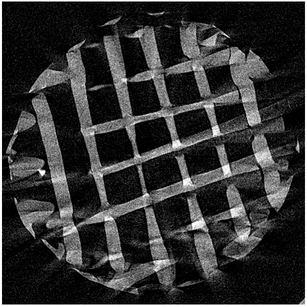


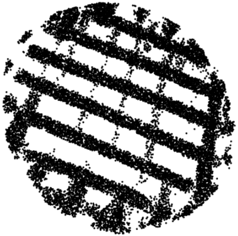


Set estimation

Point sets are a fundamental source of information in many disciplines:

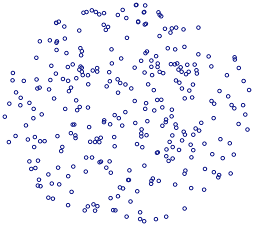
- samples from probability distributions
- portions of realizations of point processes observed within bounded sampling windows
- have substantive meaning (for example, landmarks in industrial parts or in biological structures)





Set estimation

- The term **set estimation** refers to the statistical problem of estimating an unknown set S from a random sample of points $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ whose distribution is closely related to S.



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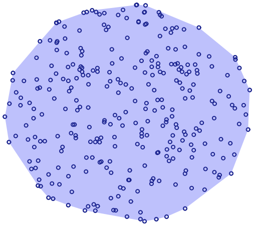

Set estimation

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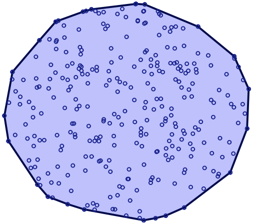
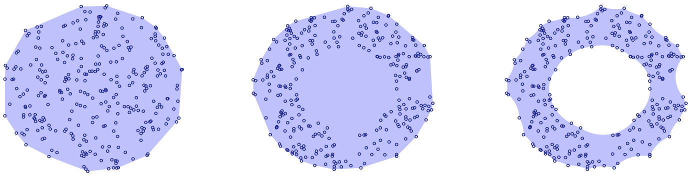
- Apart from the set itself, we may be interested in approximating a particular characteristic of the set, the length in $\mathbb{R}^2$ or surface area in general dimension $\mathbb{R}^d$



Set estimation

- There are different geometrical structures that can capture the shape of a set from a sample of points taken from it.
- The extent to which these structures manage faithfully to reproduce the original set depends heavily on the geometrical characteristics of the set.



Set estimation

Let S be a nonempty compact subset of $\mathbb{R}^d$ and let $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ be a random sample from X, where X denotes a random variable in $\mathbb{R}^d$ with distribution P_X and support S.

- Devroye-Wise estimator: $S_n = \bigcup_{i=1}^n B(X_i, \varepsilon_n)$
- Convex hull estimator: $\mathrm{conv}(\mathcal{X}_n)$
- α-convex hull estimator: $C_\alpha(\mathcal{X}_n) = (\mathcal{X}_n \oplus \alpha \mathring{B}) \ominus \alpha \mathring{B}$

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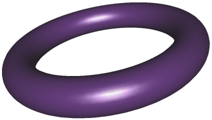
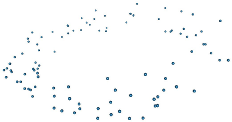
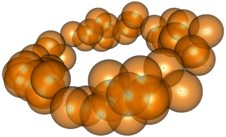

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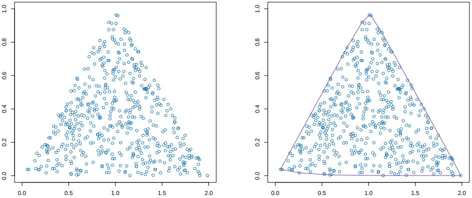

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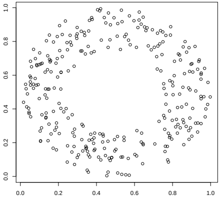
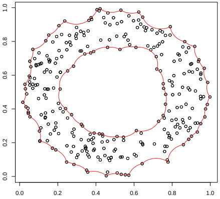




Set estimation

- The boundary of the α-convex hull comprises arcs of circles (in 2D), or spherical caps (in 3D), and intersections thereof.
- If now we approximate the boundary of the α-convex hull by a polygonal curve (in 2D) or by a polyhedral surface (in 3D), then we get another object that approximates the original set, called the α-shape.

Implementation

- Some of the set estimators in literature are easy to implement.
  - For instance, the convex hull of a set of points in d–dimensional Euclidean space can be computed using functions in package `geometry` .
- The implementation of the α-convex hull, however, is not so immediate and some effort is required in order to compute it efficiently.

Implementation

- Edelsbrunner, Kirkpatrick, and Seidel (1983) proposed an algorithm to construct the α-convex hull of a finite set of points in $\mathbb{R}^2$. The algorithm is based on the closed relationship that exists between this construct and Delaunay triangulations.
- Edelsbrunner and Mücke (1994) give the algorithm to compute the α-shape in 3D. It can be computed from the α-complex of the sample, which is a subcomplex of the Delaunay triangulation. The α-shape is the polytope (in a general sense) formed by the union of all simplices of the α-complex.

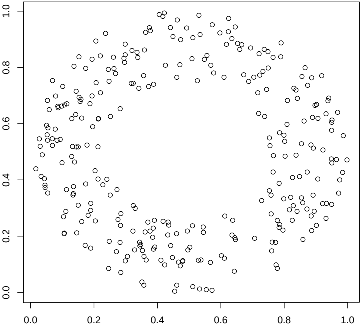
Edelsbrunner, H., Kirkpatrick, D. G., and Seidel, R. (1983)
On the shape of a set of points in the plane. *IEEE Trans. Inform. Theory*, 29(4), 551–559.

Edelsbrunner H, Mücke E. P. (1994)
Three-dimensional Alpha Shapes. *ACM Trans. Graph.*, 13(1), 43–72.

Implementation

- The R package **`alphahull`** computes the alpha-shape and alpha-convex hull of a given sample of points in the plane. The package also includes, among others, a function that returns the Delaunay triangulation and its dual Voronoi diagram and a function to calculate the Devroye-Wise estimator.
- The R package **`alphashape3d`** comprises functions to compute, represent and display the α-shape of a given sample of points in three-dimensional Euclidean space.

alphahull package: α-convex hull



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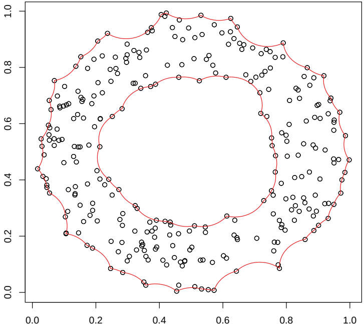



```r
> alpha <- 0.1
> ahull.obj <- ahull(x, alpha = alpha)
> plot(ahull.obj)
```

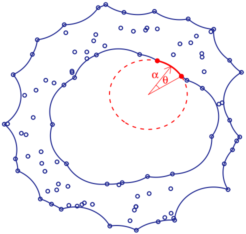
alphahull package: α-convex hull

- The boundary of $C_\alpha(\mathcal{X}_n)$ is completely determined. It is formed by the union of arcs of balls of radius α besides possible isolated points.
- We can compute the perimeter of the α-convex hull.

  The extremes of an arc can be written as $c + \alpha A_\theta(v)$ and $c + \alpha A_{-\theta}(v)$, bring $A_\theta(v)$ the clockwise rotation of angle θ of the unitary vector v. The length of each arc is $2\theta\alpha$

- We can compute the area of the α-convex hull.

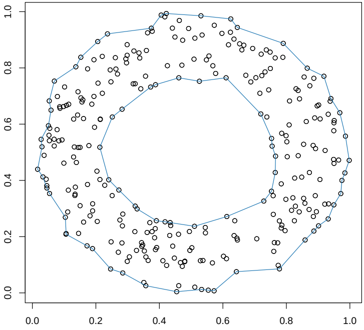
  The area of $C_\alpha(\mathcal{X}_n)$ is the sum of the areas of the connected components




```
> ahull.obj$length
```

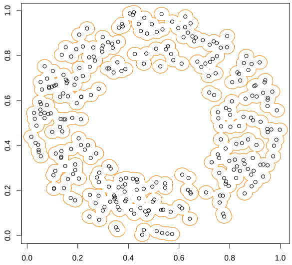
```
> areaahull(ahull.obj)
```

alphahull package: α-shape




```
> alpha <- 0.1
> ashape.obj <- ashape(x, alpha = alpha)
> plot(ashape.obj)
```

alphahull package: Devroye-Wise estimator




```
> eps <- 0.03
> dw.obj <- dw(x, eps = eps)
```

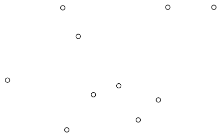
- We can compute the area of the DW estimator (not included in the library).
- The volume of the union of a family of 3D balls can be computed using the Structural Bioinformatics Library (SBL), a C++/Python API by Cazals and Dreyfus (2016).

alphahull package: Voronoi diagram and Delaunay triangulation

- The Voronoi diagram of $\mathcal{X}_n$ is a covering of the plane by n regions V_i, where for $i = 1, \dots, n$,

$$V_i = \{x \in \mathbb{R}^2 : \ \|x - X_i\| \leq \|x - X_j\| \ \text{for all } X_j \in \mathcal{X}_n\}.$$

- The Delaunay triangulation of $\mathcal{X}_n$ is defined as the straight line dual to the Voronoi diagram of $\mathcal{X}_n$, that is, there exists a straight line edge between X_i and X_j if and only if V_i and V_j are Voronoi neighbours.



```
> delvor.obj <- delvor(x)
> plot(delvor.obj)
```

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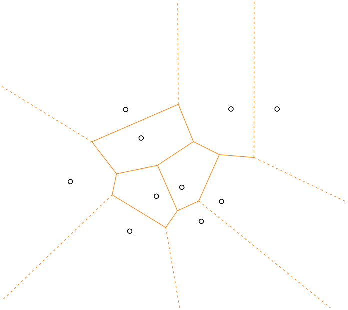

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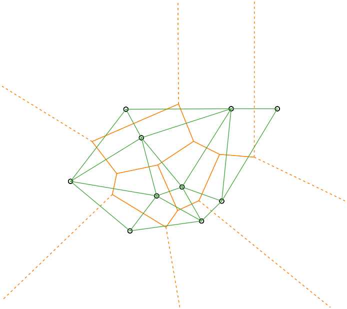

```
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```
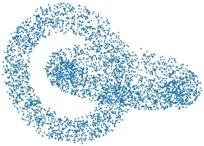
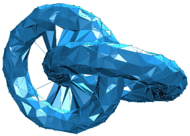
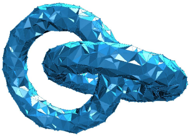

alphashape3d package: α-shape in 3D



Sample



$\alpha = 0.5$

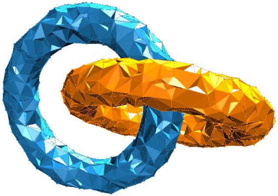


$\alpha = 0.3$

```r
> alphashape3d <- ashape3d(x, alpha = 0.5)
> plot(alphashape3d)
```

alphashape3d package: α-shape in 3D

- The library **alphashape3d** also computes the values of several attributes of the α-shape.
- We can identify the connected components of the α-shape



```
> plot(alphashape3d, byComponents = TRUE)
```

alphashape3d package: α-shape in 3D

- The function `volume_ashape` calculates the volume of the α-shape of a point cloud.
- We can also compute the volume of each connected component.
- The function `inashape3d` checks whether one or several points belong to the interior of the α- shape.
- the function `surfaceNormals` calculates the normal vectors to the triangles in the boundary of the α-shape.

Some comments on alphahull and alphashape3d

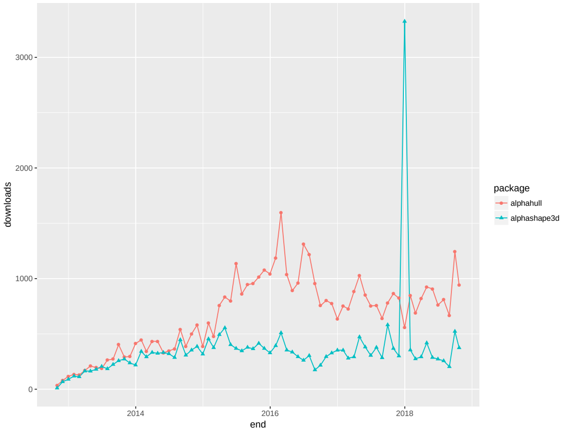
- Beatriz Pateiro-Lopez and Alberto Rodriguez-Casal (2016). alphahull: Generalization of the Convex Hull of a Sample of Points in the Plane. R package version 3.0.

Pateiro-López, B. and Rodriguez-Casal. A. (2010)
Generalizing the Convex Hull of a Sample: The R Package alphahull. *Journal of Statistical Software*, 34(5)

- Thomas Lafarge and Beatriz Pateiro-Lopez (2017). alphashape3d: Implementation of the 3D Alpha-Shape for the Reconstruction of 3D Sets from a Point Cloud. R package version 1.3.

Lafarge, T., Pateiro-López, B. , Possolo, A., Dunkers, J. P. (2014)
R implementation of a polyhedral approximation to a 3D set of points using the α-shape. *Journal of Statistical Software*, 56(4)

Some comments on alphahull and alphashape3d

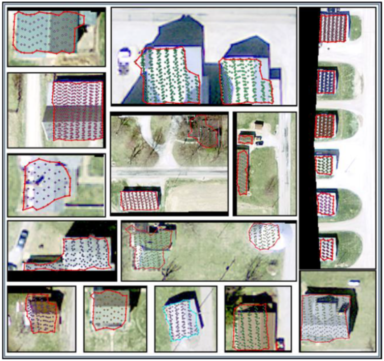


Some comments on alphahull and alphashape3d

Ural, S. et al. (2015)

Road and roadside feature extraction using imagery and Lidar data for transportation operation, *ISPRS Annals of the Photogrammetry, Remote Sensing and Spatial Information Sciences* II-3/W4

alphahull: delineation of buildings using the alpha-shape algorithm

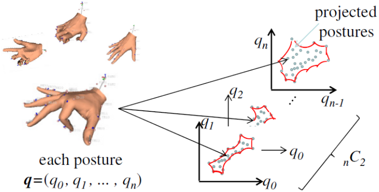


Some comments on alphahull and alphashape3d

Miyata, N. et al. (2013)

A Measuring and Analysis Method of Coupled Range of Motion of the Human Hands. *IEEE International Conference on Systems, Man, and Cybernetics*, 454, 60–69.

alphahull: estimation of the area of the range of motion in two dimensional plane

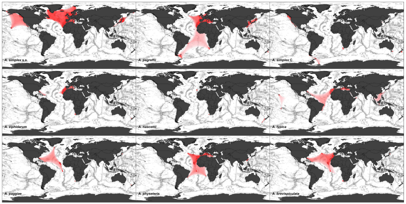


Some comments on alphahull and alphashape3d

Kuhn, T. et al. (2011)

Adaptive Radiation within Marine Anisakid Nematodes: A Zoogeographical Modeling of Cosmopolitan, Zoonotic Parasites. *PLoS ONE*, 6(12), e28642.

alphahull: species-specific distribution patterns of Anisakis spp.

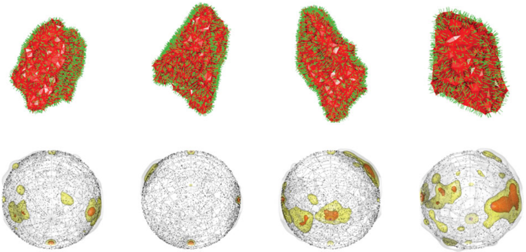


Some comments on alphahull and alphashape3d

Possolo, A. (2016)

Spatial statistics: Marks, maps, and shapes. *Quality Engineering*, 28(1), 69–90

alphashape3d: α-shapes and surface normals of cement particles (modeling and measuring the structure and properties of cement-based materials)

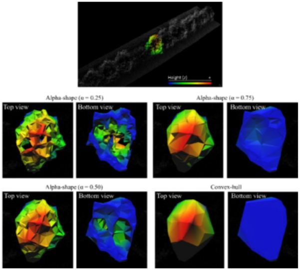


Some comments on alphahull and alphashape3d

Colaco, A. F. et al. (2017)

Orange tree canopy volume estimation by manual and LiDAR-based methods. *Advances in Animal Biosciences*, 8(2), 477–480

alphashape3d: tree canopy volume estimation using α-shapes



Thanks

Beatriz Pateiro López (beatriz.pateiro@usc.es)