CENTRO DE INVESTIGACIÓN Y TECNOLOGÍA MATEMÁTICA DE GALICIA
GALICIAN CENTRE FOR MATHEMATICAL RESEARCH AND TECHNOLOGY

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

Overall ranking in Data Envelopment Analysis with artificial units using R

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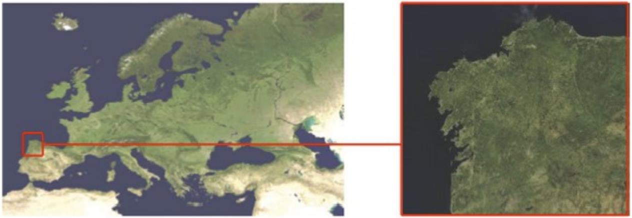
XIII XORNADA DE USUARIOS DE R EN GALICIA

The airport network of Spain



(a) Geographical location of Galicia within Europe.

(b) A Coruña Airport.

(c) Vigo Airport.

(d) Santiago Airport.

Figure: Map of Galicia and the three main airports in the region.

Task: rank the airports according to the efficiency

Assessing the efficiency of DMUs

Data environment analysis (DEA)

- It is a non-parametric methodology in Operations Research and Economics.
- The efficiency of a set of homogeneous Decision Making Units (DMUs) is evaluated.

Charnes, A., Cooper, W. W., and Rhodes, E. (1978). Measuring the efficiency of decision making units. European Journal of Operational Research, 2(6), 429–444.

Let $N = \{1, \ldots, n\}$ be a system of DMUs. Each of them is characterized by:

- m inputs. x_{ki} is the amount of input k, with $k = 1, \ldots, m$, of DMU i, for every $i \in N$.
- s outputs. y_{ki} is the amount of output k, with $k = 1, \ldots, s$, produced by DMU i, for every $i \in N$.

This problem is known as a multi-agent DEA problem and denoted by $(N; X; Y)$.

The relative efficiency score of DMU

Output-oriented DEA model with constant returns to scale

Let i_0 be a DMU.

FP

$$
\begin{array}{ll}
\max & \theta_{i_0} = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{j=1}^{m} v_j x_{j0}} \\
\text{subject to} & \\
& \frac{\sum_{r=1}^{s} u_r y_{ri}}{\sum_{j=1}^{m} v_j x_{ji}} \leq 1, \forall i = 1, \ldots, n. \\
& u_r, v_j \geq 0;\ r = 1, \ldots, s;\ j = 1, \ldots, m.
\end{array}
$$

- The relative efficiency score of DMU i_0 is given by θ_{i_0},

$$\theta_{i_0}^* = \frac{\sum_{r=1}^{s} u_r^* y_{r0}}{\sum_{j=1}^{m} v_j^* x_{j0}},$$

with $v^* = (v_1^*, v_2^*, \ldots, v_m^*)$ and $u^* = (u_1^*, u_2^*, \ldots, u_s^*)$ the optimal solution.

The relative efficiency score of DMU

Output-oriented DEA model with constant returns to scale

After some transformations, the following formulation is obtained:

MP

$$
\begin{array}{ll}
\max & \eta_{i_0} \\
\text{subject to} & \\
 & -\sum_{i\in N} y_{ri}\lambda_i + y_{ri_0}\eta_{i_0} \leq 0,\ r = 1,\ldots,s \\
 & \sum_{i\in N} x_{ji}\lambda_i \leq x_{ji_0},\ j = 1,\ldots,m \\
 & \lambda_i \geq 0,\ \forall i \in \{1,\ldots,n\} \\
 & \eta_{i_0} \in \mathbb{R}.
\end{array}
$$

The optimal values of both problems $\theta^*_{i_0}$ and $\eta^*_{i_0}$ satisfy that $\theta^*_{i_0} = \frac{1}{\eta^*_{i_0}}$.

- If a DMU is efficient, $\theta^*_{i_0} = \eta^*_{i_0} = 1$.
- A DMU is inefficient if $\theta^*_{i_0} < 1$ when Problem FP, or alternatively, if $\eta^*_{i_0} > 1$ when Problem MP.

For this purpose, we solve a LP problem using `lpsolveAPI` library.

A new approach of cooperation

- If a coalition of DMUs $S \subseteq N$ is formed, an artificial DMU $[i_S]$ is defined.
- This artificial DMU requires a certain amount of inputs to produce another amount of outputs, being both quantities obtained from the members of S.
- If $S \subseteq N$ is formed, the $\overline{f}$-$\underline{f}$ *artificial DMU* $[i_S]$ is defined by the input vector and the output vector given by

$$x_{k[i_S]} = \overline{f}(\{x_{kj}\}_{j \in S}), \text{ with } k = 1, \ldots, n,$$
$$y_{k[i_S]} = \underline{f}(\{y_{kj}\}_{j \in S}), \text{ with } k = 1, \ldots, n$$

Kritikos, M. N. (2017). A full ranking methodology in data envelopment analysis based on a set of dummy decision making units. Expert Systems with Applications, 77, 211–225.

sum-sum? min-min? min-max? max-min? max-max?

We develop specific R code for handling inputs and outputs for coalitions.

A new approach of cooperation

- As a novelty, we asses the overall efficiency of a merger of DMUs.
- That is, the merger of a certain coalition of DMUs S influences into the relative efficiency of $[i_S]$.
- We use `CoopGame` library for identifying all coalitions in N

On the cooperation of DMUs

A multi-agent DEA problem is denoted by $(N; X; Y)$.

DEA $\bar{f}$-$\underline{f}$ game

González-Díaz, J., García-Jurado, I., and Fiestras-Janeiro, M.G. (2023). An introductory course on mathematical game theory and applications. Vol. 238. American Mathematical Society.

A *DEA $\bar{f}$-$\underline{f}$ game* $(N; X; Y; \boldsymbol{e})$ (or simply, $\boldsymbol{e}$) is defined as follows:

$$\boldsymbol{e}(S) = \begin{cases} \frac{1}{\eta^*_{[i_S]}}, & \text{if } \emptyset \neq S \subseteq N, \\ 0, & \text{otherwise.} \end{cases}$$

$\eta^*_{[i_S]}$ is the optimal value of Problem 4 for the DEA problem $(N^S; X^S; Y^S)$, with coalition S as a block, for DMU $i_0 = [i_S]$.

For each coalition S, we obtain $\boldsymbol{e}(S)$ using own R code.

Properties of DEA games

Let e be a DEA $\bar{f}$-$\underline{f}$ game.

Monotonicity of DEA games

- A TU game v is said to be *increasingly monotonic* if, for each pair $S, T \subseteq N$ with $S \subset T$, we have $v(S) \leq v(T)$.
- This means that the earning of coalition of players in S increases when new players join S.

Let $(N; X; Y)$ be a multiagent DEA problem. Thus, the associated DEA min-max game $(N; X; Y; \boldsymbol{e})$ is increasingly monotone.

- In DEA min-max games, the efficiency of $S \subseteq N$ increases when merging new DMUs.
- In other cases, counterexamples can be found.

Ranking DMUs

- Task: ranking DMUs in multi-agent DEA problems using DMUs' efficiency is used as criterion.
- To this aim, values or solutions for TU games were used.

Hinojosa, M. A., Lozano, S., Borrero, D. V., and Mármol, A. M. (2017). Ranking efficient DMUs using cooperative game theory. Expert Systems with Applications, 80, 273-283.

The Shapley value of (N, v)

$$Sh_i(N,v) = \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!\,(|N|-|T|-1)!}{|N|!}(v(T\cup\{i\}) - v(T)), \text{ for every } i \in N.$$

The Banzhaf value of (N, v)

$$Bz_i(N,v) = \sum_{T \subseteq N \setminus \{i\}} \frac{1}{2^{|N|-1}}(v(T\cup\{i\}) - v(T)).$$

Both are implemented in `CoopGame` package.

Ranking DMUs: an example

A DEA problem with 12 players

DMU	Input 1	Input 2	Output 1	Output 2	Output 3	Output 4
1	17.02	5.0	42	45.3	14.2	30.1
2	16.46	4.5	39	40.1	13.0	29.8
3	11.76	6.0	26	39.6	13.8	24.5
4	10.52	4.0	22	36.0	11.3	25.0
5	9.50	3.8	21	34.2	12.0	20.4
6	4.79	5.4	10	20.1	5.0	16.5
7	6.21	6.2	14	26.5	7.0	19.7
8	11.12	6.0	25	35.9	9.0	24.7
9	3.67	8.0	4	17.4	0.1	18.1
10	8.93	7.0	16	34.3	6.5	20.6
11	17.74	7.1	43	45.6	14.0	31.1
12	14.85	6.2	27	38.7	13.8	25.4

Table: Input and output data in the example in Kritikos (2017).

Kritikos, M. N. (2017). A full ranking methodology in data envelopment analysis based on a set of dummy decision making units. Expert Systems with Applications, 77, 211–225.

Ranking DMUs: an example

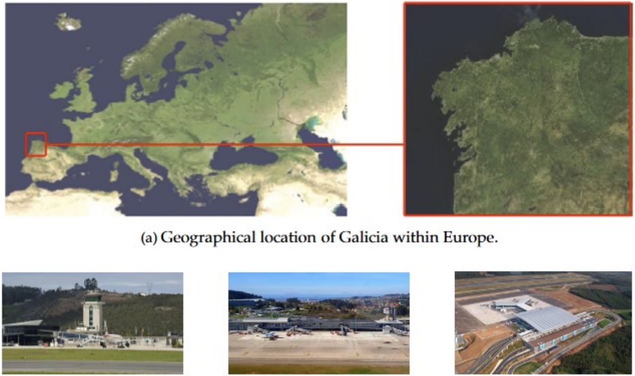
A DEA problem with 12 players

	sum-sum		min-min		min-max		max-max		max-min	
DMU	(SH)	(B)	(SH)	(B)	(SH)	(B)	(SH)	(B)	(SH)	(B)
1	0.09800	0.02114	0.08549	0.00095	0.08574	0.00064	0.11068	0.03654	0.09507	-0.01735
2	0.08795	0.00845	0.08642	0.00113	0.08574	0.00064	0.09266	0.01474	0.10391	-0.00965
3	0.08273	0.00371	0.08371	0.00076	0.08414	0.00063	0.08382	0.00433	0.10844	-0.00177
4	0.09592	0.01723	0.08696	0.00125	0.08574	0.00064	0.08353	0.00298	0.18073	0.06631
5	0.09997	0.02361	0.08696	0.00125	0.08574	0.00064	0.08909	0.00690	0.14759	0.02486
6	0.08923	0.01009	0.08696	0.00125	0.08574	0.00064	0.08385	0.00148	0.00881	-0.08675
7	0.09825	0.02176	0.08696	0.00125	0.08574	0.00064	0.08476	0.00265	0.06882	-0.02459
8	0.08016	0.00242	0.08172	0.00077	0.08224	0.00061	0.07879	0.00055	0.10599	-0.00290
9	0.07390	-0.00908	0.08696	0.00125	0.08574	0.00064	0.08152	-0.00030	-0.00989	-0.07519
10	0.06900	-0.00988	0.08156	0.00093	0.08152	0.00060	0.07677	-0.00024	0.06548	-0.01837
11	0.08392	0.00526	0.08193	0.00048	0.08421	0.00063	0.09514	0.02018	0.05764	-0.05012
12	0.04097	-0.02187	0.06437	0.00019	0.06767	0.00045	0.03938	-0.01759	0.06740	-0.01593

Table: Numerical results. The efficient DMUs (in gray), the Shapley value and the Banzhaf value for the five DEA TU games.

- **Some DMUs have a negative contribution.**

An application: the airport network of Spain



(a) Geographical location of Galicia within Europe.

(b) A Coruña Airport. (c) Vigo Airport. (d) Santiago Airport.

Figure: Map of Galicia and the three main airports in the region.

Task: rank the airports using DEA games

- We measure the capability for increasing the overall efficiency in the event of a merger.

An application: the airport network of Spain

Airport	IATA	Airport	IATA
Madrid-Barajas	MAD	Barcelona-El Prat	BCN
Palma de Mallorca	PMI	Málaga-Costa del Sol	AGP
Alicante-Elche	ALC	Gran Canaria	LPA
Tenerife Sur	TFS	Valencia	VLC
Sevilla	SVQ	Ibiza	IBZ
Lanzarote	ACE	Tenerife Norte	TFN
Bilbao	BIO	Fuerteventura	FUE
Menorca	MAH	Santiago-Rosalía de Castro	SCQ
Asturias	OVD	Zaragoza	ZAZ
Vitoria	VIT	A Coruña	LCG

Table: List of the most relevant airports of Spain.

Task: rank the airports using DEA games.

- We measure the capability for increasing the overall efficiency in the event of a merger.

An application: the airport network of Spain

	Inputs				
Airport	Runways	Terminals	Estimated Revenue (M€)	Estimated Capacity (M pax)	Estimated CO_2 (kt)
MAD	4	5	1600	80	1200
BCN	3	2	1200	65	900
PMI	2	1	500	40	500
AGP	2	3	400	30	380
ALC	1	1	300	20	280
LPA	2	1	250	18	240
TFS	1	1	220	16	210
VLC	1	1	180	15	170
SVQ	1	1	150	12	150
IBZ	1	1	140	11	140
ACE	1	1	130	10	130
TFN	1	1	110	9	110
BIO	1	1	100	8	100
FUE	1	1	95	8	95
MAH	1	1	70	6	70
SCQ	1	1	60	5	60
OVD	1	1	35	3	35
ZAZ	1	1	90	2	50
VIT	1	1	40	1.5	30
LCG	1	1	30	2	25

Table: Inputs for the most relevant airports of Spain.

Task: rank the airports using DEA games.

- We measure the capability for increasing the overall efficiency in the event of a merger.

An application: the airport network of Spain

	Outputs					
Airport	Passengers (M)	Movements (k)	Cargo (t)	Share (%)	Direct Destinations	% International
MAD	66.2	430.6	643000	23	230	60
BCN	55	348	156000	19	200	75
PMI	33.3	243.2	30000	12	170	85
AGP	24.9	187	4000	9	150	85
ALC	18.4	126.1	3000	6	120	90
LPA	15.2	142.9	20000	5	140	80
TFS	13.7	92.3	2000	5	110	85
VLC	10.8	93.1	15000	4	100	75
SVQ	9.2	73.1	9000	3	90	70
IBZ	9.1	85.1	1000	3	95	90
ACE	8.7	75	2000	3	85	85
TFN	6.8	87	1000	2	50	20
BIO	6.3	58	27000	2	80	65
FUE	6.4	53	1000	2	70	85
MAH	4.2	36	500	1	60	80
SCQ	3.6	30	3000	1	60	55
OVD	1.9	20	500	0.5	35	40
ZAZ	0.6	13	182000	0.2	20	40
VIT	0.3	9	70000	0.1	10	30
LCG	1.2	14	200	0.4	25	20

Table: Outputs for the most relevant airports of Spain.

Task: rank the airports using DEA games.

- We measure the capability for increasing the overall efficiency in the event of a merger.

An application: the airport network of Spain

	sum-sum		min-min		min-max		max-max		max-min	
DMU	(SH)	(B)	(SH)	(B)	(SH)	(B)	(SH)	(B)	(SH)	(B)
MAD	0.0545	0.0054	0.0495	$-1.2837 \cdot 10^{-4}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0503	$3.2530 \cdot 10^{-5}$	-0.0750	$-9.6598 \cdot 10^{-2}$
BCN	0.0466	-0.0016	0.0500	$2.2252 \cdot 10^{-6}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0503	$3.2530 \cdot 10^{-5}$	0.0009	$-4.9333 \cdot 10^{-2}$
PMI	0.0573	0.0114	0.0500	$1.0099 \cdot 10^{-5}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0503	$3.2530 \cdot 10^{-5}$	0.0811	$-6.6919 \cdot 10^{-4}$
AGP	0.0473	-0.0011	0.0496	$1.6975 \cdot 10^{-6}$	0.0498	$1.9703 \cdot 10^{-6}$	0.0496	$-2.1634 \cdot 10^{-5}$	0.0092	$-4.6886 \cdot 10^{-2}$
ALC	0.0535	0.0058	0.0500	$3.9098 \cdot 10^{-6}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0502	$5.4480 \cdot 10^{-6}$	0.0918	$5.6761 \cdot 10^{-3}$
LPA	0.0506	0.0010	0.0504	$4.2979 \cdot 10^{-4}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0502	$7.6015 \cdot 10^{-6}$	0.0786	$-5.2000 \cdot 10^{-5}$
TFS	0.0501	0.0013	0.0500	$2.5617 \cdot 10^{-6}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0503	$3.2530 \cdot 10^{-5}$	0.0874	$2.1978 \cdot 10^{-3}$
VLC	0.0491	-0.0003	0.0500	$4.1680 \cdot 10^{-6}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0502	$5.4480 \cdot 10^{-6}$	0.0849	$7.6117 \cdot 10^{-4}$
SVQ	0.0442	-0.0018	0.0466	$-2.0079 \cdot 10^{-5}$	0.0481	$1.8444 \cdot 10^{-6}$	0.0474	$-1.6333 \cdot 10^{-6}$	0.0783	$1.0463 \cdot 10^{-4}$
IBZ	0.0510	0.0013	0.0515	$1.6979 \cdot 10^{-3}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0502	$5.4480 \cdot 10^{-6}$	0.0930	$4.6059 \cdot 10^{-3}$
ACE	0.0535	0.0050	0.0509	$8.3850 \cdot 10^{-4}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0501	$2.9327 \cdot 10^{-5}$	0.0878	$1.1897 \cdot 10^{-3}$
TFN	0.0494	-0.0012	0.0344	$-2.0460 \cdot 10^{-2}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0502	$5.4480 \cdot 10^{-6}$	0.0271	$-3.3725 \cdot 10^{-2}$
BIO	0.0508	0.0011	0.0505	$6.0702 \cdot 10^{-4}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0502	$1.8211 \cdot 10^{-5}$	0.0796	$-4.1577 \cdot 10^{-4}$
FUE	0.0496	-0.0005	0.0511	$9.2705 \cdot 10^{-4}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0500	$2.2227 \cdot 10^{-6}$	0.0865	$5.2149 \cdot 10^{-4}$
MAH	0.0472	-0.0026	0.0517	$4.3020 \cdot 10^{-3}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0500	$1.9591 \cdot 10^{-6}$	0.0818	$-3.4522 \cdot 10^{-4}$
SCQ	0.0480	-0.0021	0.0519	$2.8609 \cdot 10^{-3}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0500	$1.9073 \cdot 10^{-6}$	0.0663	$-2.2322 \cdot 10^{-3}$
OVD	0.0472	-0.0024	0.0648	$2.2172 \cdot 10^{-2}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0500	$1.9073 \cdot 10^{-6}$	0.0409	$-1.0634 \cdot 10^{-2}$
ZAZ	0.0555	0.0081	0.0550	$8.6580 \cdot 10^{-3}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0503	$3.2530 \cdot 10^{-5}$	0.0409	$-1.5325 \cdot 10^{-2}$
VIT	0.0474	-0.0018	0.0458	$-5.3729 \cdot 10^{-3}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0503	$3.2530 \cdot 10^{-5}$	-0.0035	$-2.7496 \cdot 10^{-2}$
LCG	0.0473	-0.0025	0.0463	$-4.7482 \cdot 10^{-3}$	0.0501	$1.9918 \cdot 10^{-6}$	0.0500	$1.9073 \cdot 10^{-6}$	-0.0374	$-4.9573 \cdot 10^{-2}$

Table: Numerical results. The efficient DMUs (in gray), the Shapley value and the Banzhaf value for the five DEA TU games.

An application: the airport network of Spain

	sum-sum		min-min		min-max		max-max		max-min	
DMU	(SH)	(B)	(SH)	(B)	(SH)	(B)	(SH)	(B)	(SH)	(B)
MAD	3	4	16	17	1-18	1-19	1-6	1-6	20	20
BCN	19	14	12	14	1-18	1-19	1-6	1-6	17	18
PMI	1	1	10	10	1-18	1-19	1-6	1-6	8	11
AGP	16	12	15	15	19	1-19	19	20	16	17
ALC	5	3	13	12	1-18	1-19	7-12	10-13	2	1
LPA	8	9	9	9	1-18	1-19	7-12	9	10	8
TFS	9	6	14	13	1-18	1-19	1-6	1-6	4	3
VLC	12	10	11	11	1-18	1-19	7-12	10-13	6	5
SVQ	20	16	17	16	20	20	20	19	11	7
IBZ	6	7	5	5	1-18	1-19	7-12	10-13	1	2
ACE	4	5	7	7	1-18	1-19	13	7	3	4
TFN	11	13	20	20	1-18	1-19	7-12	10-13	15	16
BIO	7	8	8	8	1-18	1-19	7-12	8	9	10
FUE	10	11	6	6	1-18	1-19	14-18	14	5	6
MAH	18	20	4	3	1-18	1-19	14-18	15	7	9
SCQ	13	17	3	4	1-18	1-19	14-18	16-18	12	12
OVD	17	18	1	1	1-18	1-19	14-18	16-18	14	13
ZAZ	2	2	2	2	1-18	1-19	1-6	1-6	13	14
VIT	14	15	19	19	1-18	1-19	1-6	1-6	18	15
LCG	15	19	18	18	1-18	1-19	14-18	16-18	19	19

Table: Positions of DMUs in the rankings based on the numerical results.

Conclusions and further research

Concluding remarks

- In this work, we analysed the impact of aggregating the inputs and the outputs of DMUs on the overall efficiency in a merger.
- The organization of the outsiders influences the overall efficiency.
- We use some values for games to rank DMUs.
- As an application, we analyse the airport network of Spain in 2022.

In the future...

- A comparative study for several DEA models can be done.
- Using alternative values games may be of interest.
- The usage of alternative sampling methodologies in large-scale problems could be checked.

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